Scaling and local scale invariance for wetting transitions and confined interfaces

1992 J. Phys. A: Math. Gen. 25 257

(http://iopscience.iop.org/0305-4470/25/2/010)

J. Phys. A: Math. Gen. **25** (1992) 257–273. Printed in the UK

Scaling and local scale invariance for wetting transitions and confined interfaces

A O Parry

H H Wills Physics Laboratory, University of Bristol, Bristol BS8 1TL, UK

Received 25 June 1991

Abstract. We study the scaling behaviour of the magnetization profile $m(z)$ of an Ising-like magnet in a parallel-plate geometry with opposite ($h_1 = -h_2$) surface fields. In the limit of large plate separation L (and zero bulk field) $m(z)$ is a scaled function of z/L for temperatures $T_w \leqslant T < T_c$, where T_w is the critical wetting temperature of each semi-infinite surface. The exact form of $m(z)$ depends on whether $T = T_w$, $T_c > T > T_w$ or $T = T_c$. The influence of the far wall on the magnetization near one surface is long-ranged and is determined (for $T < T_c$) by wetting critical exponents. We show that the results of the capillary-wave model for a local one-point function (probability distribution, energy density) may be derived by conformally mapping the corresponding quantity defined in the semi-infinite geometry at the appropriate wetting transition. We discuss the nature of local scale invariance for wetting transitions and speculate as to why conformal invariance has application to local one-point functions.

1. Introduction

The role that thermally excited capillary-wave-like fluctuations play in determining the nature of density profiles, response functions and two-body correlation functions in fluids (or Ising magnets) at interfaces has received considerable attention from theorists (for reviews see for example Fisher 1989, Evans 1990, Forgacs *et al* 1991). This is particularly true of the well studied problem of interfacial localization (say for a pure fluid below its critical temperature and at bulk saturation chemical potential $\mu = \mu_{\rm sat}(T)$) in a weak gravitational external field $V_{\rm ext} = mgz$. The width $\xi_\perp$ of the interfacial density profile $\rho(z)$ diverges as $g \to 0$ (and dimension $d \leqslant 3$) due to the growth of long-ranged Ornstein–Zernike-like correlations parallel to the interface. Such correlations have a characteristic transverse correlation length $\xi_\parallel \propto g^{-1/2}$ for all dimensions d. Studies of continuum effective interfacial (capillary-wave) Hamiltonians yield the well known 'capillary-wave' results for the dependence of $\xi_\perp$ on $\xi_\parallel$:

$$\xi_\perp \propto \begin{cases} \xi_\parallel^{(3-d)/2} & d < 3 \\ \sqrt{\ln \xi_\parallel} & d = 3 \\ \text{finite} & d > 3. \end{cases} \tag{1}$$

The divergence of $\xi_\perp$ reflects directly the Ornstein–Zernike nature of the capillary-wave fluctuations (see the argument given by Parry 1989 and Evans 1991). This, again, is a direct reflection of the importance of fluctuations in interfacial phenomena.

In the absence of random fields or random bonds equation (1) is believed to describe correctly the interfacial broadening which takes place at interfacial unbinding (wetting) transitions where the divergence of $\xi_{\parallel}$ is sensitive to the dimensionality and type of intermolecular forces present. In recent articles (Parry 1991a, b) the author has argued that for $d < d_>$ (the upper critical dimension) the spatial dependence of the density profile (and local response and pair correlation functions) at continuous wetting transitions is also a function of $z/\xi_{\perp}$. Here z measures the distance between the two interfaces that unbind from each other. Moreover we have shown that the density (magnetization) profile, local susceptibility and pair correlation function exhibit short-distance (i.e. $z \ll \xi_{\perp}$) algebraic decay which depends directly on the values of wetting critical exponents and hence (capillary-wave) fluctuation effects.

In the present article we discuss the shape of the interfacial profile and local energy density for an interface confined in parallel plate geometry. Such a situation will arise when a fluid is confined between two parallel plates, a distance L apart say, one of which preferentially adsorbs the liquid phase while the other adsorbs the gas (Parry and Evans 1990a). The scaling properties of the density profile in such a geometry are discussed in section 3. Some of these results have been communicated earlier (Parry *et al* 1991b). Calculations based on an effective Hamiltonian model of the 2D system are presented in section 4. In section 5 we show that the results obtained in section 4 for the local energy density (and magnetization profile) follow from applying the principle of local scale invariance to wetting transitions. We discuss why conformal transformations might have some application to wetting phenomena. A summary of our results and a discussion of the shape of confined interfacial profiles in $2 \leq d < 3$ is given in section 6. To begin we recall some details of critical effects at continuous wetting transitions.

2. Critical effects at wetting transitions

Consider a semi-infinite d-dimensional Ising model whose surface spins are subject to a local magnetic field $h_1 > 0$. Suppose we impose a bulk field $h < 0$ so that spins far from the surface have a net negative magnetization. Then for subcritical temperatures one of two scenarios may be observed as $h \to 0^-$: (a) a layer of finite thickness l of upspins intrudes between the surface (located in the plane $z = 0$) and bulk spins. This corresponds to a situation of partial wetting; (b) a layer of infinite thickness of upspins forms at the surface corresponding to complete wetting.

A critical wetting transition is said to occur if at $h = 0^-$ the thickness l diverges continuously as the temperature T (or, equivalently surface field h_1) approaches some wetting temperature T_{w} (surface field h_1^{w}) from below. Such a transition is known to occur in the 2D and 3D Ising model (Abraham 1980, Binder *et al* 1986, Parry *et al* 1991a). The divergence of l is characterized by the critical exponent β_{s}

$$l \sim \tilde{\varepsilon}^{-\beta_s} \tag{2}$$

where $\tilde{\varepsilon} \equiv (T_{\mathrm{w}} - T)/T_{\mathrm{w}}$ or $\tilde{\varepsilon} \equiv |h_1 - h_1^{\mathrm{w}}|/h_1^{\mathrm{w}}$. The wetting transition may be viewed as the unbinding of the upspin–downspin ($\uparrow\!\downarrow$) *interface from the surface* at $z=0$. For $d < 3$ the width of the $\uparrow\!\downarrow$ interface diverges as $l \to \infty$

$$\xi_{\perp} \sim \tilde{\varepsilon}^{-\nu_{\perp}}. \tag{3}$$

In $d = 3$, $\xi_{\perp}$ diverges only if $T > T_{\mathrm{R}}$, the roughening temperature of the system. As alluded to in the introduction, the divergence of $\xi_{\perp}$ is associated with the build-up of capillary-wave-like fluctuations at the $\uparrow\!\downarrow$ interface. Since the transverse correlation length $\xi_\parallel$ for such fluctuations diverges as $\tilde{\varepsilon} \to 0$

$$\xi_\parallel \sim \tilde{\varepsilon}^{-\nu_\parallel} \tag{4}$$

it follows from (1) that $\nu_\perp = [(3-d)/2]\nu_\parallel$. At (and above) the transition temperature the excess free-energy per unit area $\sigma_{w\downarrow}$ is the sum of the surface upspin ($\sigma_{w\uparrow}$) and $\uparrow\!\downarrow$ ($\sigma_{\uparrow\downarrow}$) contributions. This allows the identification of a 'specific heat' exponent α_s for the transition. We write

$$\Sigma^{(s)} \equiv \sigma_{w\downarrow} - (\sigma_{w\uparrow} + \sigma_{\uparrow\downarrow})$$

which has a singular contribution

$$\Sigma^{(s)}_{\rm sing} \sim \tilde{\varepsilon}^{2-\alpha_s} \qquad h = 0 \qquad \tilde{\varepsilon} \to 0. \tag{5}$$

The behaviour of $\Sigma^{(s)}_{\rm sing}$ for $h < 0$ should be described (Nakanishi and Fisher 1982) by a scaling equation

$$\Sigma^{(s)}_{\rm sing} = \tilde{\varepsilon}^{2-\alpha_s} F(|h|\tilde{\varepsilon}^{-\Delta}) \tag{6}$$

with $\Delta \equiv 2 - \alpha_s + \beta_s$ the gap exponent. Equations (2)-(6) are equally applicable to systems with long-ranged forces as are pertinent to the description of wetting at a wall-fluid interface. Mean-field analyses (see for example Dietrich 1988) have demonstrated that a critical wetting transition is possible in fluid systems with long-ranged forces provided that the ranges of the (attractive) wall-fluid and fluid-fluid (van der Waals) potentials are the same.

The values of the critical exponents for critical wetting are sensitive to the details of the forces present and to d, the dimensionality of space. Generally we must distinguish between three fluctuation regimes. For $d > d_>$ (the upper critical dimension) mean-field (MF) theory is valid. For $d < d_>$ we distinguish between a weak-fluctuation regime (WFR) and a strong fluctuation regime (SFR) where the latter corresponds to the case where all long-ranged forces are irrelevant. In the WFR the attractive part of the interfacial binding potential (see later), constitutes a relevant operator. Further details may be found in Lipowsky and Fisher (1987).

For $d \leqslant d_>$ (<3) scaling and exponent relations suggest that $l \sim \xi_\perp$ so that the $\uparrow\!\downarrow$ interface broadens (delocalizes) and depins with the same critical exponent. Explicit calculation has confirmed this prediction for $d = 2$ except for an anomalous subregime at the WFR/SFR boundary (Lipowsky and Nieuwenhuizen 1988). In the light of this general observation we have proposed a scaling ansatz for the magnetization (or density) profile near a wetting transition (Parry 1991a, b)

$$m(z) = m_0 \Xi(z\tilde{\varepsilon}^{\beta_s}, h\tilde{\varepsilon}^{-\Delta}) + m_{\rm SR}(z). \tag{7}$$

The short-ranged contribution $m_{\rm SR}(z)$ is expected to be small (away from the bulk critical temperature T_c) $\forall z$ and vanish rapidly as z increases away from the wall. The scaling function $\Xi(x, y)$ is normalized so that $\Xi(0,0) = +1$ and $\Xi(\infty, 0) = -1$. m_0 is the bulk magnetization. Hereafter we work with the limit $h = 0^-$ only so that $m_0 = m_+$ ($\equiv -m_-$). The behaviour of the magnetization profile for 'short-distances' $z \ll \tilde{\varepsilon}^{-\beta_s}$ from the surface may be determined using thermodynamic and universality arguments and depends upon the fluctuation regime describing the transition. In the SFR we expect (Parry 1991a) (ignoring $m_{\rm SR}(z)$)

$$\frac{m(z) - m_0}{m_0} \sim -\tilde{\varepsilon}^{1-\alpha_s} z^{(1-\alpha_s)/\beta_s} \qquad z\tilde{\varepsilon}^{\beta_s} \ll 1 \qquad h = 0^- \tag{8a}$$

while in the WFR (Parry 1991b)

$$\frac{m(z)-m_0}{m_0} \sim -\tilde{\varepsilon} z^{(d+1)/(3-d)} \qquad z\tilde{\varepsilon}^{\beta_s} \ll 1 \qquad h=0^- . \tag{8b}$$

Away from $T = T_c$ such short-distance expansions should be valid provided $z \gg a$ few lattice spacings.

In $d=2$ and in the absence of irrelevant operators it is easy to show using a capillary-wave Hamiltonian that (7) and (8) are obeyed exactly in the SFR and WFR (Parry 1991a, b). The algebraic behaviour of $m(z)$ in (8a) and (8b) is characterized by the short-distance expansion critical exponent θ

$$\frac{m(z)-m_0}{m_0} \sim -(z\tilde{\varepsilon}^{\beta_s})^{\theta} \qquad z\tilde{\varepsilon}^{\beta_s} \ll 1 \qquad h=0^- . \tag{9}$$

The exponent θ is universal in the SFR and WFR but may be non-universal at the boundary between the fluctuation regimes (Parry 1991b).

The above formalism is equally applicable to the complete wetting phase transition. This is the continuous phase transition from partial to complete wetting that occurs for $T_c > T > T_w$ as $h \to 0^-$, i.e. from off bulk-coexistence. We can define exponents analogous to equations (2)–(5), i.e.

$$l \sim |h|^{-\beta_s^{co}} \tag{10a}$$

$$\xi_\perp \sim |h|^{-\nu_\perp^{co}} \tag{10b}$$

$$\xi_\parallel \sim |h|^{-\nu_\parallel^{co}} \tag{10c}$$

$$\Sigma_{sing}^{(s)} \sim |h|^{2-\alpha_s^{co}} . \tag{10d}$$

The field h always constitutes a relevant scaling field so there is no SFR for the complete wetting transition. For $d < d_>^{co}$ (the upper critical dimension for complete wetting) the transition is described by the WFR and $l \sim \xi_\perp$. The critical exponents are known exactly for this case (Lipowsky 1985): $\beta_s^{co} = \nu_\perp^{co} = (d+1)/(3-d)$ and recall $\beta_s^{co} = [(3-d)/2]\nu_\parallel^{co}$ and $2-\alpha_s^{co} = (d-1)\nu_\parallel^{co}$. We expect the magnetization profile to be described by a scaling equation analogous to (7):

$$m(z) = m_0 \Xi^{co}(z|h|^{\beta_s^{co}}) + m_{SR}(z) \tag{11}$$

with a short-distance expansion near the wall

$$\frac{m(z)-m_0}{m_0} \sim -(z|h|^{\beta_s^{co}})^{\theta} \qquad zh^{\beta_s^{co}} \ll 1 \tag{12a}$$

with exponent (Parry 1991a)

$$\theta = \frac{d+1}{3-d} . \tag{12b}$$

Equations (11) and (12) have been verified in $d=2$ using a capillary-wave model (Parry 1991a).

3. Phase equilibria and magnetization profiles in the asymmetric slab

We now consider a d-dimensional Ising magnet that is of infinite extent in $(d-1)$ dimensions but is of finite width L in the z-direction. Allow for local surface fields h_1

(>0) and h_2 which couple to the plane of spins at $z=0$ and $z=L$ respectively. Hereafter we consider the case of perfect asymmetry

$$h_1=-h_2$$

only. We will refer to this as the $+-$ geometry and begin by recalling the details of the phase equilibria for such a system.

We choose our surface layer spin coupling (surface enhancement) so that each semi-infinite surface (i.e. $L=\infty$) undergoes a critical wetting transition as $T\to T_w^-$ and $h=0$. Then, in the $+-$ geometry, criticality (or pseudo-criticality) of the confined system is predicted to occur at some temperature $T_{c,L}$ that is determined for large L by T_w (Parry and Evans 1990a). For $T<T_{c,L}$ two phases coexist at $h=0$ and their magnetization profiles correspond to a film of upspins at one wall and a film of downspins at the other respectively. As $T\to T_{c,L}$ the films grow in thickness and the difference in the excess magnetization diminishes and vanishes at $T=T_{c,L}$. The associated criticality is conjectured to belong to the $(d-1)$ dimensional Ising model universality class. For $T\geqslant T_w$ there is only one phase in the $+-$ geometry $\forall L$†. The presence of true symmetry breaking in the two semi-infinite geometries ($L=\infty$) leads to a very large correlation length ($\xi_\parallel\sim L^{2/(3-d)}(d<3)$ and $\xi_\parallel\sim e^L(d\geqslant 3)$) in the temperature range $T_c>T>T_w$ (Parry and Evans 1990a, b). On the basis of finite-size scaling arguments it was argued that $T_{c,L}$ is shifted below T_w (for $L\to\infty$) by an amount determined by the wetting critical exponent β_s

$$T_{c,L}-T_w\propto -L^{-1/\beta_s} \qquad L\to\infty \tag{13}$$

where the constant of proportionality may be temperature dependent. For wetting temperatures very close to the bulk-critical temperature T_c the location of $\tilde{t}_{c,L}\equiv(T_c-T_{c,L})/T_c$ is, at least in MF theory, consistent with the scaling hypothesis (Parry and Evans 1991b, Swift *et al* 1991, Indeku *et al* 1991)

$$\tilde{t}_{c,L}=L^{-1/\nu}Y(h_1L^{-\Delta_1/\nu}) \qquad +-. \tag{14}$$

Here ν and Δ_1 are the bulk correlation length and surface critical gap exponents respectively. In the following we shall assume that h_1 is chosen so that T_w is well below the bulk critical temperature T_c.

For $T_c>T\geqslant T_w$ the magnetization profile ($\forall L$ and $h=0$) resembles a ⇅ interface located at the centre $z=L/2$ of the slab. The precise shape of the profile $m(z)$, for asymptotically large L, depends on whether $T=T_w$ or $T>T_w$ reflecting the fluctuation effects in the SFR and WFR respectively. In order to see this consider the finite-size scaling of the surface excess free-energy $\Sigma^{(s)}$, regarding L as an extra scaling variable. For dimension $d<3$ and short-ranged forces we consider a real space renormalization group transformation, in the vicinity of the critical wetting fixed point, which rescales the lattice anisotropically. If the scale factor is $b_\parallel$ parallel to the wall then it is $b_\perp=b_\parallel^{(3-d)/2}$ normal to the wall (Lipowsky and Fisher 1987). We expect that $\Sigma^{(s)}$ contains a singular piece which rescales in the standard way as a homogeneous function

$$\Sigma^{(s)}_{\text{sing}}(\tilde{\varepsilon},h,L)=b_\parallel^{-(d-1)}\Sigma^{(s)}_{\text{sing}}(b_\parallel^{y_\varepsilon}\tilde{\varepsilon},\,b_\parallel^{y_h}h,\,L/b_\perp) \tag{15}$$

with temperature and magnetic eigenvalues

$$y_\varepsilon=1/\nu_\parallel \qquad y_h=\Delta/\nu_\parallel$$

† For dimension $d=3$ we assume that $T_R<T_w$ so that specific lattice effects may be ignored (see Parry and Evans 1991a).

consistent with the scaling equation (6). Setting $b_\perp \propto L$ we find for $T \sim T_w$

$$\Sigma^{(s)}_{\text{sing}} = L^{-\tau}\tilde{W}(L\tilde{\varepsilon}^{\beta_s}, h\tilde{\varepsilon}^{-\Delta}). \tag{16}$$

The exponent $\tau = 2(d-1)/(3-d)$, which follows from making use of the hyperscaling relation $2-\alpha_s = (d-1)\nu_\parallel$ and $\beta_s = [(3-d)/2]\nu_\parallel$ (Parry and Evans 1990a). The arguments of the scaling function $\tilde{W}$ are consistent with the shift (13). Differentiating (16) w.r.t. $\tilde{\varepsilon}$ yields the singular contribution to the surface layer magnetization. In particular for $\tilde{\varepsilon} = h = 0$ we find (Parry *et al* 1991b)

$$m_1^{\text{sing}} \sim L^{-\tau+1/\beta_s} \qquad h=0 \qquad T=T_w. \tag{17}$$

The result (17) may also be interpreted as the surface layer perturbation $\Delta m_1^{\text{sing}}(L) \equiv m_1(L) - m_1(\infty)$. $\Delta m_1(L)$ measures the perturbation to the magnetization at one wall due to the finite size (i.e. presence of the other wall). In $d=2$ this perturbation is particularly long-ranged. For $d=2$ we have $\tau=2$, $\beta_s=1$ and so $\Delta m_1^{\text{sing}} \propto L^{-1}$ which should be compared with the Fisher–deGennes result $\Delta m_1^{\text{sing}} \propto L^{-d}$ valid exactly at bulk criticality $T=T_c$ and $h=0$ (Au-Yang and Fisher 1980). Thus the critical wetting surface perturbation is longer-ranged in this dimension. Using the known numerical results for β_s in $d<3$ (Lipowsky and Fisher 1987) we conclude that the critical wetting perturbation (17) is longer-ranged than the Fisher–deGennes result $\forall d \leqslant 2.3$. For $d \to 3^-$ the exponent in (17) rapidly diverges to $-\infty$, implying an exponentially small surface layer perturbation (i.e. short-ranged) in $d=3$.

The prediction for the surface layer perturbation can be generalized to distances z away from the walls. This amounts to a straightforward generalization of the scaling of the magnetization at critical wetting transitions reviewed in section 2. For $+-$ boundary conditions we make the ansatz

$$\frac{m(z)}{m_0} = M(z\tilde{\varepsilon}^{\beta_s}, h\tilde{\varepsilon}^{-\Delta}, L\tilde{\varepsilon}^{\beta_s}) \qquad T \leqslant T_w \tag{18}$$

valid asympotically for $\varepsilon \to 0$, $h \to 0$ and $L \to \infty$. We have ignored any short-ranged contributions to $m(z)$. Combining (18) and (17) we predict that exactly at $T=T_w$ $(h=0)$ $m(z)$ is a function of the single variable z/L with an algebraic decay

$$\frac{m(z)-m_0}{m_0} \sim -C_s(d)\left(\frac{z}{L}\right)^{\tau-1/\beta_s} \qquad T=T_w \qquad h=0 \qquad \frac{z}{L} \ll 1 \qquad \text{SFR} \tag{19}$$

close to one wall. This result should hold throughout the SFR. Note that since the non-universal metric factors of $z\tilde{\varepsilon}^{\beta_s}$ and $L\tilde{\varepsilon}^{\beta_s}$ in M are the *same* the critical amplitude ratio $C_s(d)$ is universal in the SFR.

Similarly for temperatures $T_c > T > T_w$ we expect the surface layer perturbation Δm_1 to be long-ranged at $h=0$ in the WFR reflecting the fact that $m(z)$ is a (different) scaled function of z/L. Using the same arguments as for critical wetting we expect that $\Sigma^{(s)}$ contains a singular scaling contribution for large L

$$\Sigma^{(s)}_{\text{sing}} = L^{-\tau}\tilde{W}^{\text{co}}(Lh^{\beta_s^{\text{co}}}) \qquad \text{WFR} \tag{20}$$

valid in the WFR; for short-ranged forces this implies $d<3$. The argument of the scaling function represents the ratio of relevant length scales, $l \sim |h|^{-\beta_s^{\text{co}}}$ and $L/2$, measured perpendicular to the walls. On the basis of (20) we expect that

$$\xi_\parallel \sim L^{2/(3-d)}\Lambda^{\text{co}}(h^{(3-d)/(d+1)}L) \qquad d<3 \tag{21}$$

for short-ranged forces, where we have used the result $\beta_s^{\text{co}} = (3-d)/(d+1)$ in the WFR.

To calculate $\Delta m_1(L)$ we recall that in the WFR the singular contribution to the isothermal susceptibility $(\partial m_1/\partial h)_T$ is finite (Parry 1991a, b). Therefore in the strip we expect a scaling form

$$\left(\frac{\partial m_1}{\partial h}\right)_T = X^{\mathrm{co}}(h^{(3-d)/(d+1)}L) \qquad d < d_>^{\mathrm{co}} \tag{22}$$

which implies the singular surface perturbation

$$\Delta m_1 \propto L^{-[(d+1)/(3-d)]} \qquad h = 0 \tag{23}$$

for $T_c > T > T_w$ in systems with short-ranged forces in $d < 3$. The same result holds throughout the WFR. The perturbation is clearly long-ranged, although it is easy to show that it is not as long-ranged as the perturbation at the wetting temperature (17). The result (23) may be generalized to distances away from the wall by introducing an additional scaling variable $Lh^{(3-d)/(d+1)}$ into the ansatz (11) for the profile at complete wetting. For $h=0$, $m(z)$ is a scaled function of z/L with

$$\frac{m(z)-m_0}{m_0} \sim -C_w(d)\left(\frac{z}{L}\right)^{(d+1)/(3-d)} \qquad d < d_>^{\mathrm{co}} \qquad h = 0 \tag{24}$$

for short-distances $z \ll L$. Again, the critical amplitude ratio $C_w(d)$ should be universal in the WFR ($d < d_>^{\mathrm{co}}$). Equations (19) and (24) may be summarized by the algebraic law

$$\frac{m(z)-m_0}{m_0} \sim -\left(\frac{z}{L}\right)^{\theta} \qquad h = 0 \qquad \frac{z}{L} \ll 1$$

with the short-distance expansion exponent θ taking its SFR value (8*a*) and WFR value (8*b*) for $T = T_w$ and $T_c > T > T_w$ respectively.

Implicit in the above analyses is the assumption that the effects of bulk criticality may be ignored. The short-distance expansions (8), (9), (12*a*), (19) and (24) are valid for distances $\xi_b \ll z \ll L$ where ξ_b is the bulk correlation length. Near bulk criticality the divergence of ξ_b implies a different short-distance expansion for distances $a \ll z \ll \xi_b$ where a is a lattice spacing. Clearly, exactly at $T = T_c$ the algebraic decay of $m(z)$ (at a single wall) is completely determined by bulk critical fluctuations and wetting has no effect. The scaling of the magnetization profile in the $+-$ geometry at bulk criticality is a well known consequence of finite-size scaling (see e.g. Diehl 1986). In $d = 2$ conformal invariance predicts the full form of $m(z)$ in the $+-$ geometry for distances z away from the wall (Burkhardt and Xue 1991)

$$m(z) \propto \left[\frac{L}{\pi}\sin\frac{\pi z}{L}\right]^{-1/8} \cos\frac{\pi z}{L} \qquad T = T_c \qquad h = 0 \qquad +-.$$

We shall return to the subject of conformal invariance in section 5.

4. Calculations of the magnetization profile and energy density in models of the 2D +− strip

To study the magnetization profile, local energy density and correlation length in the $d = 2$ $+-$ strip we use the well known continuum capillary-wave Hamiltonian

$$H = \int_{-\infty}^{+\infty} \mathrm{d}x \left[\frac{\sigma}{2}\left(\frac{\mathrm{d}l(x)}{\mathrm{d}x}\right)^2 + U(l(x))\right] \tag{25}$$

with σ the $\uparrow\downarrow$ interfacial stiffness coefficient and $U(l)$ the binding potential modelling the direct interaction between the unbinding interfaces. To model fluctuations in the strip for systems with short-ranged forces we choose

$$U(l)=\begin{cases}\infty & l<0 \qquad l>L\\ -U & 0<l<R \qquad L-R<l<L\\ 0 & \text{otherwise}\end{cases} \tag{26}$$

with $U>0$, in the vicinity of $T=T_w$. For $T>T_w$ we need not include an attractive potential well at each wall to model the asymptotic $L\to\infty$ behaviour. Rather we include a bulk field h to examine the scaling predictions (20)-(24) and write

$$U(l)=\begin{cases}\infty & l<0 \qquad l>L\\ hl & \text{otherwise}\end{cases} \tag{27}$$

with $h>0$. In order to solve (25) for general $U(l)$ it is convenient to study the Schrödinger equation formulation of the path integral problem (for a discussion see e.g. Lipowsky 1985). Statistical averages may be written in terms of the eigenfunctions ψ_n and eigenvalues E_n of the equation

$$\left(-\frac{1}{2\sigma\beta^2}\frac{d^2}{dl^2}+U(l)-E_n\right)\psi_n(l)=0 \tag{28}$$

with $\beta=1/(k_BT)$. In particular we can associate $\Sigma^{(s)}$ with the ground state energy

$$\Sigma^{(s)}=E_0 \tag{29}$$

and the correlation length $\xi_\parallel$ with

$$\xi_\parallel=\frac{k_BT}{E_1-E_0}. \tag{30}$$

The magnetization profile is constructed in the usual solid-on-solid way by assuming that the graph $l(x)$, describing the instantaneous position of the $\uparrow\downarrow$ interface, separates regions of bulk upspin magnetization $m_0=m_+$ (for $z<l(x)$ say) from regions of bulk downspin (for $l(x)<z$). The probability of finding the interface (at position x along the wall) between 'heights' $l(x)$ and $l(x)+dl$ is

$$P(l)\,dl=\psi_0^2(l)\,dl \tag{31}$$

and is translationally invariant in the x-direction. The magnetization and energy density then follow as

$$m(z)=m_0\left(1-2\int_0^z\psi_0^2(l)\,dl\right) \tag{32}$$

and

$$\varepsilon(z)\propto\psi_0^2(z) \tag{33}$$

respectively. The energy density is clearly a local quantity. To proceed we consider the cases $T\sim T_w$ and $T>T_w$ separately.

4.1. $T \sim T_w$

From Kroll and Lipowsky (1983) the wetting temperature for the model defined with the potential

$$U(l)=\begin{cases}\infty & l<0\\ -U & 0<l<R\\ 0 & \text{otherwise}\end{cases}$$

satisfies

$$\beta_w=\sqrt{\frac{\pi^2}{8R^2U\sigma}}. \tag{34}$$

The ground-state energy E_0 of the confined system is easily found by setting $\psi_0(0)=\psi_0(L)=0$ and matching $\psi_0(l)$ and its derivatives at $R^\pm$ and $(L-R)^\pm$. The leading-order behaviour of E_0 is

$$E_0=\frac{1}{4\sigma\beta_w^2L^2}G_0(\gamma) \tag{35}$$

where γ is the dimensionless scaled variable

$$\gamma=c\frac{L}{R}\tilde{\varepsilon} \tag{36}$$

with $\tilde{\varepsilon}=(\beta-\beta_w)/\beta_w$ and c a pure number which is O(1). The scaling function $G_0(\gamma)$ satisfies the implicit equations

$$\gamma=-\sqrt{G_0}\tanh\sqrt{G_0/2} \qquad \gamma\leqslant 0 \tag{37}$$

$$\gamma=-\sqrt{-G_0}\tan\sqrt{-G_0/2} \qquad \gamma\geqslant 0. \tag{38}$$

Clearly the singular contribution to the free-energy E_0 has the required scaling form (16); recall $\tau=2$ and $\beta_s=1$ in $d=2$.

The scaling function G_0 is identical to that found in the restricted sos model of the same problem (Privman and Svrakíc 1988). Details of the higher eigenvalues E_n may be found in that article. Exactly at the wetting temperature the eigenvalues satisfy

$$E_n=\frac{n^2\pi^2}{2\sigma\beta_w^2(L-2R)^2} \qquad n=0,1,2 \tag{39}$$

which implies that the ground-state energy is exactly zero at $T=T_w$, i.e. the energy is precisely the same as a free ($U(l)=0\ \forall l$) interface. In addition at $T=T_w$, the symmetric ground-state wavefunction is particularly simple

$$\psi_0(l)=\begin{cases}\dfrac{1}{\sqrt{L-R}}\sin\dfrac{\pi l}{2R} & 0<l<R\\[2ex] \dfrac{1}{\sqrt{L-R}} & R<l<\dfrac{L}{2}.\end{cases} \tag{40}$$

Neglecting the variation of $\psi_0(l)$ close to the walls it follows that the interface wanders 'freely' in the strip. The probability of finding the interface at a given height l is dependent of l. For later purposes it is convenient to define a rescaled probability distribution

$$\tilde{P}(l)\equiv P(l)/P(R) \tag{41}$$

so that in the present case

$$\tilde{P}(l)=1 \qquad T=T_{\mathrm{w}} \qquad h=0 \qquad R<l<L-R. \tag{42}$$

The reason for introducing $\tilde{P}(l)$ is that this quantity exists for both critical and complete wetting transitions (at single walls $L=\infty$) exactly at the transition temperature (and $h=0$), while $P(l)$ vanishes at finite l due to the normalization condition on $\psi_0(l)$.

Neglecting the variation of $\psi_0(l)$ near the walls it follows from (31)-(33) and (40) that the profile $m(z)$ and energy density $\varepsilon(z)$ satisfy

$$m(z)\sim m_0\left(1-\frac{2z}{L}\right) \qquad T=T_{\mathrm{w}} \qquad h=0 \tag{43}$$

and

$$\varepsilon(z)\propto\frac{1}{L} \qquad T=T_{\mathrm{w}} \qquad h=0 \tag{44}$$

respectively. Similarly the correlation length $\xi_\parallel$ exactly at the wetting temperature is from (30) and (39)

$$\xi_\parallel\sim\frac{2\sigma\beta_{\mathrm{w}}}{\pi^2}L^2 \qquad T=T_{\mathrm{w}} \qquad h=0 \tag{45}$$

where we have neglected terms which are $O(R/L)$. The magnetization profile (43) is clearly a scaled function of z/L, vindicating the scaling hypothesis (18). For short distances $R<z\ll L$

$$\frac{m(z)-m_0}{m_0}=-\frac{2z}{L}$$

in agreement with the scaling prediction (19). The critical amplitude ratio $C_s(2)=2$. Furthermore, simple perturbation theory demonstrates that this short-distance expansion is unaffected by including long-ranged irrelevant operators in the binding potential $U(l)$. The amplitude is therefore universal in the SFR as expected.

4.2. $T_c>T>T_w$

Provided $|L\tilde{\varepsilon}|\gg 1$ the temperature (surface-field) dependence of $U(l)$ may be neglected and the potential (27) suffices to study the large L behaviour of the model. We set $2\beta^2\sigma\equiv 1$ for convenience. The wavefunctions $\psi_n(l)$ are simply Airy functions:

$$\psi_n(l)=A\cdot \mathrm{Ai}(h^{1/3}l-h^{-2/3}E_n(h,L))+B\cdot \mathrm{Bi}(h^{1/3}l-h^{-2/3}E_n(h,l)) \tag{46}$$

where we have written $E_n=E_n(h,L)$ to emphasize the field dependence of the eigenvalues.

From the boundary conditions $\psi_n(0)=\psi_n(L)=0$ it follows that

$$\frac{\mathrm{Ai}(-\tilde{W}_n^{\mathrm{co}}(h,L))}{\mathrm{Bi}(-\tilde{W}_n^{\mathrm{co}}(h,L))}=\frac{\mathrm{Ai}(h^{1/3}L-\tilde{W}_n^{\mathrm{co}}(h,L))}{\mathrm{Bi}(h^{1/3}L-\tilde{W}_n^{\mathrm{co}}(h,L))} \tag{47}$$

where

$$E_n=h^{2/3}\tilde{W}_n^{\mathrm{co}}(h,L)$$

and we have maintained the notation of equation (20). From (47) it follows that E_n have the scaling form

$$E_n = h^{2/3}\tilde{W}_n^{\mathrm{co}}(h^{1/3}L) \tag{48}$$

so that $\Sigma^{(s)}$ and $\xi_\parallel$ have precisely the conjectured scaling forms (see (20) and (21)). The scaling functions have the asymptotic behaviour

$$\tilde{W}_n^{\mathrm{co}}(z) \to \frac{n^2\pi^2}{z^2} - \frac{z^2}{2} + \ldots \qquad z \to 0$$

$$\tilde{W}_n^{\mathrm{co}}(z) \to |\lambda_n| - \frac{\mathrm{Bi}(\lambda_n)}{\mathrm{Ai}(\lambda_n)'}\,\mathrm{e}^{-4/3(z)^{3/2}} + \ldots \qquad z \to \infty$$

where λ_n are the zeros of the Airy function Ai.

Exactly at $h = 0$ the correlation length is (reinserting the factor of $2\beta^2\sigma$)

$$\xi_\parallel = \frac{2\sigma\beta L^2}{3\pi^2} \qquad h = 0 \qquad T > T_\mathrm{w} \qquad L \to \infty. \tag{49}$$

The cross-over in $\xi_\parallel$ from its behaviour exactly at $T = T_\mathrm{w}$ (equation (45)) to equation (49) is described by a scaling function whose argument is $L\tilde{\varepsilon}$ (see Privman and Svrakíc 1988). Equation (49) is valid $\forall T > T_\mathrm{w}$ (fixed) in the limit $L \to \infty$. It follows that the ratio of $\xi_\parallel$ exactly at T_w to its value for $T \gtrsim T_\mathrm{w}$ is universal in the limit $L \to \infty$ ($h = 0$)

$$\frac{\xi_\parallel(T = T_\mathrm{w})}{\xi_\parallel(T \gtrsim T_\mathrm{w})} = 3 \qquad L \to \infty. \tag{50}$$

The ground-state wavefunction $\psi_0(l)$, exactly at $h = 0$, is

$$\psi_0(l) = \sqrt{\frac{2}{L}}\sin\frac{\pi l}{L} \qquad h = 0 \qquad T > T_\mathrm{w} \qquad 0 < l < L \tag{51}$$

so that the profile and energy density satisfy

$$m(z) = m_0\left(1 - \frac{2z}{L} + \frac{1}{\pi}\sin\frac{2\pi z}{L}\right) \qquad h = 0 \qquad T > T_\mathrm{w} \tag{52}$$

and

$$\varepsilon(z) \propto \frac{2}{L}\sin^2\frac{\pi z}{L} \qquad h = 0 \qquad T > T_\mathrm{w} \tag{53}$$

respectively. The rescaled probability distribution is defined (cf 41) by

$$\tilde{P}(l) \equiv P(l)/P(1) \tag{54}$$

where we assume that $L \gg 1$. Clearly we have

$$\tilde{P}(l) \approx \frac{L^2}{\pi^2}\sin^2\frac{\pi l}{L} \qquad h = 0 \qquad T > T_\mathrm{w}. \tag{55}$$

The magnetization profile (52) is clearly a function of the scaling variable (z/L). For $z \ll L$ we expand $m(z)$ and find

$$\frac{m(z) - m_0}{m_0} \sim -\frac{4\pi^2}{3}\left(\frac{z}{L}\right)^3 \tag{56}$$

in agreement with the scaling prediction (24). The critical amplitude ratio takes the universal value $C_w(2) = 4\pi^2/3$ throughout the WFR. This result may be established explicitly by considering the effect of irrelevant long-range operators in $U(l)$ using elementary perturbation theory.

5. Local scale invariance for wetting

For a correlation function at bulk criticality measuring the expectation value of products of local operators much has been learnt from Polyakov's (1970) suggestion that the scaling covariance of the correlation function is obeyed for spatially dependent rescaling factors $b(\boldsymbol{r})$ which locally preserve the lattice structure, i.e. angle preserving, conformal transformations. In $d = 2$ the conformal group is infinitely large and severely constrains the nature of criticality in this dimension. Alternatively, rather than exploiting conformal invariance as a symmetry, conformal mappings can be used to relate critical behaviour in different geometries (see for example the collected papers in Cardy 1988). A conformal mapping can be used, for example, to derive the magnetization profile in finite-width Ising strips (with a variety of boundary conditions) at $T = T_c$ and $h = 0$ from knowledge of the magnetization profile in the semi-infinite Ising model (Burkhardt and Xue 1991).

In contrast, global scale invariance at a continuous wetting transition requires an essential anisotropic lattice rescaling. In $d = 2$ $b_\perp = \sqrt{b_\parallel}$ (see section 3) so that the global transformation does not preserve angles. If one were to apply the principle of local scale invariance of local operators at wetting transitions this anisotropy must be taken into account. Suppose, for example, that at $T = T_w$ and $h = 0$ (in the semi-infinite Ising model) we could define a two-point correlation function $O(z_1, z_2; R) \equiv \langle\phi(\boldsymbol{r}_1)\phi(\boldsymbol{r}_2)\rangle - \langle\phi(\boldsymbol{r}_1)\rangle\langle\phi(\boldsymbol{r}_2)\rangle$, where R is the parallel separation of the two points $\boldsymbol{r}_1$ and $\boldsymbol{r}_2$. Under a global rescaling we would expect 0 to transform according to

$$O(z_1, z_2; R) = b_\perp^{-2x_\phi} O\left(\frac{z_1}{b_\perp}, \frac{z_2}{b_\perp}; \frac{R}{b_\parallel}\right) \tag{57}$$

with x_ϕ the scaling dimension of the local operator ϕ. If we were to generalize (57) to exploit the expected local scale invariance of $O(z_1, z_2; R)$ it is clear that a conformal transformation is inappropriate since the local rescaling must necessarily be $b_\perp(\boldsymbol{r}) = \sqrt{b_\parallel(\boldsymbol{r})}$. The same applies to higher-point functions.

For one-point functions at wetting transitions, however, the situation is somewhat simpler. Under a global rescaling the one-point function $p(z_1) = \langle\phi(\boldsymbol{r}_1)\rangle$ transforms as

$$p(z_1) = b_\perp^{-x_\phi} p\left(\frac{z_1}{b_\perp}\right) \tag{58}$$

so that a knowledge of $b_\parallel$ is inessential. If we were to exploit the local scale invariance of $p(z_1)$ there may be some critical systems and dimensionalities where our choice of $b_\parallel(\boldsymbol{r})$ is irrelevant provided that the transformed geometry is also translationally invariant in the parallel direction. If this were not the case then p would depend on a set of length scales $\{R_i\}$ which would transform as $\{R_i/b_\parallel(\boldsymbol{r})\}$. Clearly if we are to avoid the enormous complication of taking into account the anisotropic nature of the rescaling we require that the geometries being mapped should exhibit translational invariance in $d-1$ dimensions. With this proviso we are free to use conformal mappings to exploit the local scale invariance of one-point functions at wetting transitions. Clearly the necessity of maintaining translational invariance is extremely restrictive in our choice of conformal mapping. The most obvious choice, and the one pertinent to the present analysis, is the well known logarithmic mapping

$$w(z) = \frac{L}{\pi} \ln z \tag{59}$$

which maps the semi-infinite plane $z = x + iy$ ($y > 0$) into the strip $w = u + iv$ (with $0 < v < L$). In context of the capillary wave model we choose the rescaled probability distribution $\tilde{P}(l)$ (see (41)) as a local operator. From (9), (32) and (58) we identify the exponent $x_\phi = \theta - 1$. For short-ranged forces ($h = 0$) $\tilde{P}(l)$ may be calculated for the SFR critical wetting and WFR complete wetting phase transitions (Parry 1991a). These correspond to temperatures $T = T_w$ and $T > T_w$ respectively. For the purpose of the mapping we write $\tilde{P}(l) \equiv \tilde{P}(x, y)$, meaning the (rescaled) probability of finding the interface (at x along the wall) at height y. The results for the semi-infinite plane (calculated using the capillary-wave model) are

$$\tilde{P}(x, y) = 1 \qquad T = T_w \qquad h = 0 \tag{60}$$

and

$$\tilde{P}(x, y) = y^2 \qquad T_c > T > T_w \qquad h = 0. \tag{61}$$

These results follow naturally from the short-distance expansion for $d = 2$ critical wetting (8*a*) and complete wetting (12) respectively. Applying the logarithmic mapping (59) to (61) and (62) generates the rescaled probability $\tilde{P}(u, v)$ in the $+-$ strip: we find

$$\tilde{P}(u, v) = 1 \qquad T = T_w \qquad h = 0 \qquad 0 < v < L \tag{62}$$

and

$$\tilde{P}(u, v) = \frac{L^2}{\pi^2} \sin^2 \frac{\pi v}{L} \qquad T_c > T > T_w \qquad h = 0 \qquad 0 < v < L \tag{63}$$

which are in precise agreement with the explicit capillary-wave results (42) and (55). From (62) and (63) the energy-density and magnetization profile follow immediately. The present analysis is in the same spirit as that of Burkhardt and Eisenreigler (1985) who applied the mapping (59) to calculate $m(z)$ at $T = T_c$, $h = 0$ in the strip (with $h_1 = h_2$) from the algebraic decay law for the profile in the semi-infinite system: $m(z) \sim z^{-\beta/\nu}$ (Fisher and deGennes 1978).

Before ending this section we discuss the application of the mapping (59) to 2D wetting transitions which may be found in systems with long-ranged forces. We do not expect the mapping to be applicable to systems which belong to the MF regime. This follows from noting that there is no short-distance expansion (since $l \gg \xi_\perp$) in the MF fluctuation regime so that the simple homogeneous law (58) does not hold. In addition, the two results (62) and (63) may be regarded as pertinent to the SFR and WFR respectively in the absence of irrelevant operators (recall from (8*b*) that θ is universal in the WFR). This leaves us with the SFR/WFR and MF/WFR intermediate fluctuation regimes (Lipowsky and Fisher 1987). These correspond (in $d = 2$) to the case where an algebraic tail l^{-2} constitutes a marginal operator in the binding potential $U(l)$ (Kroll and Lipowsky 1983, Lipowsky and Nieuwenhuizen 1988). In order to sit exactly at $T = T_w$ (and $h = 0$) we need to specify the global nature of $U(l)$. This is straightforward for the MF/WFR borderline. We simply specify that there is no relevant attractive short-ranged operator in $U(l)$, i.e.

$$U(l) = \begin{cases} \infty & l < 0 \\ \tilde{\omega}/l^2 & l > 0 \end{cases} \qquad T = T_w \qquad \text{MF/WFR} \tag{64}$$

with $\tilde{\omega}$ a strength parameter $\tilde{\omega}>0$. The short-distance expansion exponent θ (and hence $\tilde{P}(l)$) are known for this model (Parry 1991b). θ is non-universal

$$\theta = 2+\sqrt{1+8\tilde{\omega}\sigma\beta^2} \tag{65}$$

so that

$$\tilde{P}(l) = l^{1+\sqrt{1+8\tilde{\omega}\sigma\beta^2}} \qquad T=T_w \qquad h=0. \tag{66}$$

Hereafter we set $2\sigma\beta^2 \equiv 1$. Applying the conformal mapping (59) to (66) implies that

$$\psi_0(l) = (\sin \pi l/L)^{1/2(1+\sqrt{1+4\tilde{\omega}})}/(L\pi^{-1/2}\Gamma(1+\tfrac{1}{2}\sqrt{1+4\tilde{\omega}}))^{1/2} \tag{67}$$

is the ground-state wavefunction for the strip potential

$$U(l) = \begin{cases} \infty & l<0 \qquad l>L \\ \dfrac{\tilde{\omega}\pi^2}{L^2 \sin^2 \pi l/L} & \text{otherwise.} \end{cases} \tag{68}$$

Note that we have applied the same mapping to the semi-infinite one-body binding potential. Substitution shows that (67) is indeed the exact ground-state wavefunction for the potential (68) with energy

$$E = \frac{\pi^2}{4L^2}(1+\sqrt{1+4\tilde{\omega}})^2. \tag{69}$$

It is remarkable that the prediction of local scale invariance remains valid even in the presence of a long-ranged marginal operator.

The situation at the SFR/WFR is more complicated. The details of the transition are highly sensitive to the short-ranged structure of $U(l)$. Even in the simplest case where $U(l)$ has a purely repulsive contribution (at $l=1$ say) the short-distance expansion for $\tilde{P}(l)$ has logarithmic corrections (Parry 1991b) and the generalization of (58) for local scale transformations is problematic. This subject requires further research.

6. Conclusion

In the present article we have discussed the role that (capillary-wave-like) fluctuations play in determining the nature of magnetization profiles in parallel plate geometries with opposite surface fields. We have argued that such fluctuations lead to:

(i) Scaling of the profile $m(z)$ (and energy density). The profile is a scaled function of the variable z/L for $T=T_w$ and $T_c>T>T_w$ $(h=0)$.

(ii) Algebraic decay law for $m(z)$ near a surface. As a consequence of this algebraic behaviour the surface perturbation $\Delta m_l(L)$ is long-ranged provided d is below the upper critical dimension of the wetting transition. For $d \leqslant 2.3$ the surface perturbation at $T=T_w$ is larger than the Fisher–deGennes effect at $T=T_c$.

(iii) Local scale invariance for systems with short-ranged forces. For one-point functions a logarithmic conformal mapping reproduces the strip $(+-)$ geometry results in $d=2$ for both $T=T_w$ and $T_c>T>T_w$.

It is important to recognize the universal nature of the short-distance expansion for $m(z)$ in the $+-$ geometry. Here we concentrate on the predictions for a 2D system. The expansions (19) and (24) are specific to the SFR and WFR respectively. Thus, the same universal behaviour should be observed if a 'real' 2D fluid (or phase-separated binary mixture) with long-ranged dispersion foces were confined between parallel walls that are wet by different phases. For practical purposes it is of course highly unlikely that a system may be found where each phase undergoes a wetting transition at exactly the same temperature. The universal result (19) with $C_s(2)=2$ (or rather its equivalent for a binary mixture) would therefore seem to be of academic interest only. The specifications for observing the scaling law (24) are, however, not nearly so restrictive. For instance we may forgo the requirement of perfect asymmetry. Rather we stipulate that the temperature be above the wetting temperature of each wall. That is, for a binary mixture say, we require that $T>T_{wA}$, T_{wB} where T_{wA} is the temperature above which the A-rich phase wets the wall at $z=0$ (say) and T_{wB} is the temperature above which the wall at $z=L$ is completely wet by the B-rich phase. Provided the chemical potentials are at saturation value we expect the universal short-distance expansion for the concentration $C_A(z)$ of species A

$$\frac{C_A(z)-C_A}{\Delta C_A}=\frac{-2\pi^2}{3}\left(\frac{z}{L}\right)^3+\dots \qquad \xi_b \ll z \ll L$$

where ΔC_A is the difference in the concentration of species A in the two phases, i.e. $\Delta C_A = C_A^{(1)} - C_A^{(2)} (>0)$.

Similarly if one could find a 2D pure fluid where the liquid (l) and gas (g) phases wet the walls at $z=0$ and $z=L$ for $T>T_w$ and $T>T_d$ respectively, then we expect the universal density expansion

$$\frac{\rho(z)-\rho_l}{\rho_l-\rho_g}=\frac{-2\pi^2}{3}\left(\frac{z}{L}\right)^3+\dots \qquad \xi_b \ll z \ll L$$

for $T_c>T>T_w$, T_d.

The conclusions (i) and (ii) may be extended to yield a semiquantitative theory for $m(z)$ for arbitrary dimensionality $d<3$ for systems with short-ranged forces. We assume that in the confined geometry for $T_c>T\geq T_w$ and $h=0$, $m(z)$ is a scaled function of z/L (in the limit $L\to\infty$) whose algebraic form is dominated by the short-distance behaviour near the wall. If for the semi-infinite wetting problem $P(l)\propto l^{\theta-1}$ near the wall, then in the $+-$ geometry we suppose

$$P(l)\alpha\left(\frac{l}{L}\right)^{\theta-1}\left(1-\frac{l}{L}\right)^{\theta-1}. \tag{70}$$

This may be integrated (as in (32)) to yield

$$\frac{m(z)}{m_0}\sim 1-2I_{z/L}\left(\frac{1-\alpha_s}{\beta_s},\frac{1-\alpha_s}{\beta_s}\right) \qquad T=T_w \qquad h=0 \tag{71}$$

and

$$\frac{m(z)}{m_0}\sim 1-2I_{z/L}\left(\frac{d+1}{3-d},\frac{d+1}{3-d}\right) \qquad T_c>T>T_w \qquad h=0 \tag{72}$$

for confinement at and above the critical wetting temperature respectively. Here $I_x(a, b)$ is the normalized incomplete beta function. These approximate forms are guaranteed to yield the correct short-distance exponent for $0<z\ll L$. They do not yield accurate corrections to leading-order behaviour; hopefully these are small. Their degree of success may be gauged by comparison with the explicit capillary-wave results in $d=2$ (cf (43) and (52)). Remarkably, we find that (in $d=2$) (71) is identical to the capillary-wave result (43), while (72) is numerically rather close to (52) for all z. In fact for

$T_c > T > T_w$

$$\frac{|m(z)^{\text{beta}} - m(z)^{\text{capillary-wave}}|}{m_0} < 0.03$$

in an obvious notation. Even if the error is measured relative to $m(z)^{\text{capillary-wave}}$ we find accuracy to better than 6%. With these observations we conclude that the beta function approximation is at least a good semiquantitative guide to $m(z)$.

It is interesting to consider the behaviour of $m(z)^{\text{beta}}$ for $T = T_w$ and $T_c > T > T_w$ as $d \to 3^-$. As this (upper critical) dimension is approached the interfacial region becomes strongly localized to $z = L/2$. In fact as $d \to 3^-$ the scaled midpoint gradient $(m^{\text{beta}}(L/2))'/Lm_0$ (which is independent of L) diverges like

$$\frac{m^{\text{beta}}(L/2)'}{Lm_0} \sim (3-d)^{-1/2} \qquad (73)$$

for both $T = T_w$ and $T_c > T > T_w$. In deriving (78) for $T = T_w$ we have made use of the result

$$\frac{1-\alpha_s}{\beta_s} \sim \frac{4}{3-d}(1 + \mathrm{O}(3-d)^{2/3}) \qquad d \to 3^-$$

valid for critical wetting with short-ranged forces (David and Leibler 1990). The divergence of the scaled midpoint gradient is consistent with the idea (Parry and Evans 1990b) that for both critical and complete wetting transitions the critical amplitude ratio $l/\xi_\perp \sim (3-d)^{-1/2}$ as $d \to 3^-$ (short-ranged forces). This provides a mechanism for a smooth cross-over to $d = d_> = 3$ behaviour where $l \sim \xi_\perp^2$ for continuous wetting transitions.

Acknowledgments

The author has benefited from conversations with R Evans and D B Abraham. This research was supported by the SERC.

References

Abraham D B 1980 *Phys. Rev. Lett.* **44** 1165
Au-Yang H and Fisher M E 1980 *Phys. Rev.* B **21** 3956
Binder K, Landau D P and Kroll D M 1985 *Phys. Rev. Lett.* **56** 2272
Burkhardt T W and Eisenreigler E 1985 *J. Phys. A: Math. Gen.* **18** L83
Burkhardt T W and Xue T 1991 *Phys. Rev. Lett.* **66** 895
Cardy J L (ed) 1988 *Finite-Size Scaling* (Amsterdam: North-Holland)
David F and Leibler S 1990 *Phys. Rev.* B **41** 12926
Diehl H W 1986 *Phase Transitions and Critical Phenomena* vol 10, ed C Domb and J Lebowitz (London: Academic) p 75
Dietrich S 1988 *Phase Transitions and Critical Phenomena* vol 12, ed C Domb and J Lebowitz (London: Academic) p 1
Evans R 1990 *Liquids at Interfaces* (Les Houches Session XLVIII) ed J Charvolin, F J Joanny and J Zinn-Justin (Amsterdam: Elsevier) p 1
—— 1991 *Inhomogeneous Fluids* ed D Henderson (New York: Dekker) ch 5
Fisher M E 1989 *Statistical Mechanics of Membranes and Surfaces. Jerusalem Winter School for Theoretical Physics* vol 5 (Singapore: World Scientific) p 19
Fisher M E and deGennes P G 1978 *C. R. Acad. Sci. Paris* B **287** 207

Forgacs G, Lipowsky R and Nieuwenhuizen T M 1991 *Phase Transitions and Critical Phenomena* vol 14, ed C Domb and J Lebowitz (London: Academic)
Indeku J O, Owczarek A L and Swift M R 1991 *Phys. Rev. Lett.* **66** 2174
Kroll D M and Lipowsky R 1983 *Phys. Rev.* B **28** 5273
Lipowsky R 1985 *Phys. Rev.* B **32** 1731
Lipowsky R and Fisher M E 1987 *Phys. Rev.* B **36** 2126
Lipowsky R and Nieuwenhuizen T M 1988 *J. Phys. A: Math. Gen.* **21** L89
Nakanishi H and Fisher M E 1982 *Phys. Rev. Lett.* **49** 1565
Parry A O 1989 *PhD thesis* (unpublished)
—— 1991a *J. Phys. A: Math. Gen.* **24** 1335
—— 1991b *J. Phys. A: Math. Gen.* **24** L699
Parry A O and Evans R 1990a *Phys. Rev. Lett.* **64** 439
—— 1990b *J. Phys.: Condens. Matter* **2** 7687
—— 1991a *Phys. Rev. Lett.* **66** 2175
—— 1991b (to appear in *Physica* A)
Parry A O, Evans R and Binder K 1991a *Phys. Rev.* B **43** 11535
Parry A O, Evans R and Nicolaides D 1991b *Preprint*
Polyakov A M 1970 *JETP Lett.* **12** 381
Privman V and Svrakíc N M 1988 *Phys. Rev.* B **37** 3713
Swift M R, Owczarek A L and Indeku J O 1991 *Europhys. Lett.* **14** 475